Enskog-Like Kinetic Models for Vehicular Traffic

A. Klar[1] and R. Wegener[2]

In the present paper a general criticism of kinetic equations for vehicular traffic is given. The necessity of introducing an Enskog-type correction into these equations is shown. An Enskog-like kinetic traffic flow equation is presented and fluid dynamic equations are derived. This derivation yields new coefficients for the standard fluid dynamic equations of vehicular traffic. Numerical simulations for inhomogeneous traffic flow situations are shown together with a comparison between kinetic and fluid dynamic models.

KEY WORDS: Traffic flow; Enskog equation; Boltzmann equation; fluid dynamic models; Payne equation; simulation of inhomogeneous traffic flow.

1. INTRODUCTION

Traditionally there have been three types of approaches to the modeling of traffic flow phenomena. The first and most basic one concerns microscopic or follow-the-leader models, modeling the actual response of single cars to their predecessor (see, e.g., refs. 1–3). Macroscopic models based on fluid dynamic equations have been proposed by a large number of authors (see, e.g., refs. 4–8). However, some of these models have been a subject of considerable controversy concerning their validity and applicability to traffic flow. Kinetic or Boltzmann-like models may present an intermediate step between the above two types of models. On the one hand they can be derived from microscopic considerations. On the other hand, fluid dynamic models can be derived from kinetic traffic models, as has been shown, e.g., in refs. 9–12. The first kinetic traffic models were published by Prigogine and Andrews,[13] who introduced a Boltzmann-like term to account for

---

[1] Fachbereich Mathematik, Universität Kaiserslautern, Kaiserslautern, Germany.
[2] Institut für Techno- und Wirtschaftsmathematik, Kaiserslautern, Germany.

the slowing-down interactions (see Prigogine and Herman[9] for a review). However, they and most of their successors (see, e.g., refs. 14, 11, and 10) treated the acceleration of the cars by means of a heuristic relaxation term. Only recently Nelson[15] succeeded in obtaining reasonable kinetic equations by using a kinetic description also for acceleration. In ref. 16 a new model based on Nelson's approach is described and numerically investigated for space-homogeneous traffic flow situations. For a survey concentrating on kinetic models see ref. 17.

In this paper we show that standard kinetic models are not able to deal with inhomogeneous traffic flow situations. Based on the model in ref. 16 we introduce an Enskog-type correction term into the kinetic equation. This yields a model which is able to describe correctly inhomogeneous traffic flow patterns. Moreover, we derive from the kinetic model a new fluid dynamic traffic flow equation by considerations well known from Enskog's theory of a dense gas. In particular, the coefficients in these fluid dynamic equations, such as the traffic pressure, the anticipation term, and the relaxation time, are derived in this way from the kinetic model.

Up to now simulations of inhomogeneous real-life traffic flow situations have only been done using microscopic and fluid dynamic models. The main advantage of fluid dynamic models is the small computation time necessary for such a simulation. Here one should mention a special type of microscopic model, the cellular automaton models with simulation times that are similar to those of fluid dynamic models (see, for example, refs. 18 and 19). The present paper contains numerical simulations of inhomogeneous traffic situations for the kinetic and the derived fluid dynamic equations. Concerning the computation times for the kinetic equation, it should be noted that kinetic models contain more variables than fluid dynamic models and fewer variables than microscopic ones. Therefore, one expects that computation times for the simulation of kinetic models range between those for microscopic and those for fluid dynamic models. We present a fast numerical scheme for kinetic models allowing the simulation of inhomogeneous traffic situations with reasonable computation times. Moreover, the coefficients for the fluid model are numerically evaluated for a special kinetic model. The resulting fluid equations are solved and the solutions are compared to those of the kinetic equation.

The paper is organized in the following way: In Section 2 a general criticism of kinetic equations concerning their applicability for inhomogeneous traffic flow situations is given. Section 3 contains an Enskog-type kinetic model avoiding the difficulties mentioned in Section 2. In Section 4 a derivation of fluid dynamic equations is shown yielding an Enskog correction to standard fluid dynamic models. Finally, in Section 5 we describe a fast numerical scheme to solve the kinetic equation and present

numerical results for kinetic and fluid equations for the case of a highway with a reduction of lanes from three to two.

2. CRITICISM OF STANDARD KINETIC EQUATIONS FOR VEHICULAR TRAFFIC

The basic quantity in a kinetic modeling of vehicular traffic flow is the phase-space distribution function $f(x, v, t)$. It describes the number of vehicles at x and t with speed $v \in [0, w]$, where w denotes the maximal speed of the vehicles. The evolution of f is given by a kinetic equation of the form

$$f_t + vf_x = C(f)(x, v, t) \tag{1}$$

with suitable initial and boundary values. C is an operator acting on the speed distribution function f and depending explicitly on the speed v, but on x and t only via $f(x, v, t)$. For a more detailed description of traffic flow on highways a multilane model describing each lane by its own distribution function f_α, $\alpha = 1,..., N_L$ (N_L: number of lanes) is appropriate. However, for the sake of simplicity we restrict ourselves in this paper to a cumulative treatment of these distribution functions, i.e., $f = \sum_{\alpha=1}^{N_L} f_\alpha$. Classical kinetic traffic flow theories follow this strategy as well. We mention some examples:

Prigogine and Herman[9] modeled the interactions between the cars by a slowing-down term and a relaxation term:

$$C(f) = S(f) + R(f)$$

The slowing-down term

$$S(f)(x, v, t) = (1 - P)\, f(x, v, t) \int_0^w (v' - v)\, f(x, v', t)\, dv'$$

describes the breaking-down interactions of a vehicle and depends on the probability of overtaking P. This term is derived in a similar fashion to the usual derivation of the Boltzmann equation in rarefied gas dynamics. Usually, P was assumed to depend on the density

$$\rho(x, t) = \int_0^w f(x, v, t)\, dv$$

in a linear way,[9]

$$P(\rho) = 1 - \frac{\rho}{\rho_m}$$

where ρ_m denotes the maximal density. The introduction of a term like $P(\rho)$ represents the above-mentioned cumulative treatment of the multilane effects. The relaxation term

$$R(f)(x, v, t) = -\frac{f(x, v, t) - f_0(x, v, t)}{T}$$

takes into account the acceleration of the vehicles to their desired speed. The desired speed distribution f_0 was assumed to be of the form

$$f_0(x, v, t) = \rho(x, t) F_0(v)$$

where F_0 is a given function not depending on t. The relaxation time T for the acceleration interactions was also assumed to depend on ρ

$$T(\rho) = \frac{\rho}{\rho_m - \rho}$$

Another widely used model is due to Paveri-Fontana,[11] who introduced a distribution function depending also on the desired speeds of the drivers. The following argument is not changed by doing this. We mention here also the model in ref. 16. It will be used as a basis for the Enskog-type model which we will develop in Section 3.

All these models have up to now mainly been used to simulate homogeneous traffic flow situations. Stationary distribution functions and fundamental diagrams, i.e., flux–density relations, have been computed. However, the influence of the streaming term has never been examined in detail. If the above equations are used for the description of inhomogeneous traffic flow situations, a serious drawback appears due to the positivity of the velocities v: There is no mechanism in the equations to allow perturbations to propagate backward in the negative-x direction. This can be seen by the following trivial argument: Considering a full space problem, the integral form of the kinetic equation is

$$f(x, v, t) = f(x - vt, v, 0) + \int_0^t C(f)(x + v(t' - t), v, t') \, dt'$$

In particular, this shows that the distribution function at x and t depends only on the distribution function at the values $x' \leqslant x$, $t' \leqslant t$, since v is positive. A perturbation cannot propagate backward in the negative x direction. In particular, traffic jams occurring for dense traffic situations are not allowed to travel backward. This is in striking contrast to real traffic flow observations. The above type of kinetic equation is therefore only

applicable for dilute traffic flow without backward-propagating informa-
tion. We remark that the situation for kinetic equations in gas dynamics,
such as the Boltzmann equation, is completely different. There the argu-
ment is obviously not true, since the velocity can assume positive and
negative values.

The above remark has consequences for the connection between
kinetic and fluid dynamic traffic flow equations. Formally, the fluid
dynamic equations can be derived from the kinetic ones by a moment
procedure. However, all the derived fluid dynamic equations, such as, for
example, the Lighthill–Witham, Payne, or other equations, are able to
describe backward-propagating disturbances; see ref. 4 for the basic model.
This means that one cannot expect to obtain a strict derivation of fluid
dynamic equations from the above kinetic equations due to their com-
pletely different behavior in the above-described situation.

In general, to describe correctly the behavior of dense traffic with a
kinetic equation and to obtain a consistent derivation of fluid dynamic
equations, it is necessary to include the effects of the finite size extension of
the vehicles. This can be done as shown in the next section in analogy to
Enskog's theory for a dense gas. One obtains in this way a kinetic equation
with the possibility of backward propagating disturbances. Here one
should also mention ref. 8, where the finite size requirement of the vehicles
was heuristically included in a macroscopic equation.

3. ENSKOG-LIKE VEHICULAR TRAFFIC EQUATIONS

In this section we describe an Enskog-type approach for kinetic
vehicular traffic equations on the basis of the kinetic model developed in
ref. 16. The procedure can obviously be adapted also for any other types
of kinetic equations, such as the ones mentioned in the last section.

3.1. Microscopic Model

Our kinetic model is based on the following microscopic model. Con-
sider a car 1 at place x_1 with the velocity v_1 and its leading car 2 at x_2 with
velocity v_2. Car 1 is assumed to change its velocity only in response to the
leading car. If the vehicle is faster than the leading car and the headway to
the leading car is becoming smaller than a certain threshold, its driver
slows down or passes the leading vehicle. If the vehicle is slower than the
leading car and the headway to the leading car is becoming larger than
another threshold, the car accelerates. Actually several different thresholds
for slowing down and acceleration procedures are known. One observes,
for example, thresholds based on a certain desired distance or on a minimal

acceptable distance between a driver and the leading vehicle.[2, 20] The thresholds in general depend on the velocity of the car and on the velocity of the leading vehicle. Let N be the number of thresholds under consideration. If car 1 is crossing a threshold $i \in \{1,..., N\}$, i.e., if the headway $h = x_2 - x_1$ is becoming larger of smaller than a certain threshold $h = H_i(v_1, v_2)$, then car 1 changes its velocity to the new velocity v. Velocities v_1 and v_2 are, for each threshold i, our of a certain range Ω_i of values. The new velocity is taken on instantaneously in accordance with a certain distribution function

$$\sigma_i(v, v_1, v_2; \rho), \qquad (v_1, v_2) \in \Omega_i$$

Since σ_i is a density function, it has to fulfill

$$\int_0^w \sigma_i(v, v_1, v_2; \rho) \, dv = 1$$

The dependence of σ_i on the local density ρ allows us to include the cumulative consideration of the multilane effects in the model. Moreover, σ_i may also be a function of location-dependent road condition parameters, e.g., the number of lanes; see the example in Section 5.

A slowing-down maneuver could be given, e.g., by a function $H_i(v_1, v_2) = h(v_2)$, where h represents the minimal acceptable distance to a leading vehicle with speed v_2. Since a car slows down only if its velocity is larger than that of the leading vehicle, Ω_i is in this case given by $\{(v_1, v_2); v_1 > v_2\}$. In particular, one can recover the Prigogine slowing-down term by setting $H_i = 0$, σ_i equal to

$$\sigma_i(v, v_1, v_2; \rho) = P(\rho) \, \delta(v_1 - v) + (1 - P(\rho)) \, \delta(v_2 - v)$$

and $\Omega_i = \{(v_1, v_2); v_1 > v_2\}$.

3.2. Kinetic Equation

One can now derive by arguments similiar to the derivation of the Boltzmann equation in gas dynamics[21, 22] a kinetic equation as follows: Let $f(x, v, t)$ denote as before the phase-space density. Writing down the equation for the change of the total number of vehicles leads as usual to the kinetic equation:

$$f_t + v f_x = \left(\frac{\delta f}{\delta t}\right)_g - \left(\frac{\delta f}{\delta t}\right)_l$$

where $t \in [0, \infty)$, $x \in [0, L]$, $v \in [0, w]$, and $(\delta f/\delta t)_g$ and $(\delta f/\delta t)_l$ are gain and loss terms due to discontinuous velocity changes, respectively. We can write them as

$$\left(\frac{\delta f}{\delta t}\right)_g = \sum_{i=1}^{N} \left(\frac{\delta f}{\delta t}\right)_g^i, \qquad \left(\frac{\delta f}{\delta t}\right)_l = \sum_{i=1}^{N} \left(\frac{\delta f}{\delta t}\right)_l^i$$

where $(\delta f/\delta t)_g^i$ and $(\delta f/\delta t)_l^i$ describe the gain and loss terms, respectively, due to the ith threshold. We mention that for the numerical simulations in Section 5 we will restrict consideration to the case of two thresholds corresponding to acceleration and slowing-down interactions. The gain and loss terms are given respectively by

$$\left(\frac{\delta f}{\delta t}\right)_g = \sum_{i=1}^{N} \int_{(v_1, v_2) \in \Omega_i} |v_1 - v_2| \, \sigma_i(v, v_1, v_2; \rho)$$

$$\times f_2(x, v_1, x + H_i(v_1, v_2), v_2, t) \, dv_1 \, dv_2$$

$$\left(\frac{\delta f}{\delta t}\right)_l = \sum_{i=1}^{N} \int_{(v, v_2) \in \Omega_i} |v - v_2| \, f_2(x, v, x + H_i(v, v_2), v_2, t) \, dv_2$$

Here $f_2(x, v_1, x + d, v_2, t)$ denotes the pair distribution function of the cars and leading cars at x and $x + d$, respectively and

$$\rho = \rho(x, t) = \int_0^{w} f(x, v, t) \, dv$$

To obtain from these equations a closed equation for f we have to express f_2 by f. We mention here the work of Nelson,[15] who introduced a correlation model for kinetic vehicular traffic equations using an assumption which he termed the "generalized vehicular chaos assumption." A modified vehicular chaos assumption can be stated as

$$f_2(x, v_1, x + d, v_2, t) = f(x, v_1, t) \, f(x + d, v_2, t) \, \tilde{k}(x, d, t)$$

In addition, we assume that $\tilde{k}$ does not depend explicitly on x and t, but only via the local density. Then f_2 has a form similar to Enskog's theory of a dense gas:

$$f_2(x, v_1, x + d, v_2, t) = f(x, v_1, t) \, f(x + d, v_2, t) \, k(d, \rho(x, t))$$

We mention that in Enskog's theory k depends on $\rho(x + d/2)$. In the revised Enskog theory it is a general functional of $\rho(\cdot, t)$. See refs. 23–25 for details on Enskog equations.

Defining

$$C_i^+(f)(x, v, t)$$

$$= \int_{(v_1, v_2) \in \Omega_i} |v_1 - v_2| \, \sigma_i(v, v_1, v_2; \rho) \, k(H_i(v_1, v_2), \rho)$$

$$\times f(x, v_1, t) \, f(x + H_i(v_1, v_2), v_2, t) \, dv_1 \, dv_2$$

$$- \int_{(v, v_2) \in \Omega_i} |v - v_2| \, k(H_i(v, v_2), \rho) \, f(x, v, t) \, f(x + H_i(v, v_2), v_2, t) \, dv_2$$

we obtain the Enskog-like equation

$$f_t + v f_x = \sum_{i=1}^{N} C_i^+(f)(x, v, t) \tag{2}$$

This equation will be the basis for all further investigations. Reconsidering the arguments in Section 2, one obtains here

$$f(x, v, t) = f(x - vt, v, 0) + \int_0^t \sum_{i=1}^{N} C_i^+(f)(x + v(t' - t), v, t') \, dt'$$

In this case one observes, due to the definition of C_i^+, that the distribution function at x, t depends not only on the distribution function at $x' \leqslant x$, $t' \leqslant t$, but also on the distribution function at $x' > x$. This allows backward-propagating disturbances. In contrast to the gas dynamics case, we see that the Enskog character of the equation is essential even for the description of the most basic properties of the equation such as backward-propagating disturbances.

4. MACROSCOPIC EQUATIONS

In this section we present a derivation of a fluid dynamic model with an Enskog correction term derived from the kinetic model in the last section. In particular, we obtain density-dependent coefficients for the macroscopic equations. We reconsider the kinetic model in Section 3. For the sake of a simple notation we assume from now on that the sets Ω_i are disjoint with

$$\bigcup_{i=1}^{N} \Omega_i = [0, w]^2 \tag{3}$$

Then one can define

$$\sigma(v, v_1, v_2; \rho) = \sigma_i(v, v_1, v_2; \rho) \qquad \text{and} \qquad H(v_1, v_2) = H_i(v_1, v_2)$$

for $(v_1, v_2) \in \Omega_i$. With

$$C^+ = \sum_{i=1}^{N} C_i^+$$

we obtain the equation

$$f_t + vf_x = C^+(f)(x, v, t) \tag{4}$$

where C^+ is given by

$$C^+(f)(x, v, t)$$

$$= \int_{[0, w]^2} |v_1 - v_2| \, \sigma(v, v_1, v_2; \rho) \, k(H(v_1, v_2), \rho)$$

$$\times f(x, v_1, t) \, f(x + H(v_1, v_2), v_2, t) \, dv_1 \, dv_2$$

$$- \int_0^w |v - v_2| \, k(H(v, v_2), \rho) \, f(x, v, t) \, f(x + H(v, v_2), v_2, t) \, dv_2$$

As a first step in the derivation of macroscopic equations we have, as usual in kinetic theory, to investigate the homogeneous kinetic equation.

4.1. Equilibrium (Homogeneous, Stationary Case)

The homogeneous equation for $f = f(v, t)$ is

$$f_t = C(f)(v, t) \tag{5}$$

with

$$C(f)(v, t) = \int_{[0, w]^2} |v_1 - v_2| \, \sigma(v, v_1, v_2; \rho) \, k(H(v_1, v_2), \rho))$$

$$\times f(v_1, t) \, f(v_2, t) \, dv_1 \, dv_2$$

$$- \int_0^w |v - v_2| \, k(H(v, v_2), \rho) \, f(v, t) \, f(v_2, t) \, dv_2$$

For this kinetic equation $\rho = \int_0^w f(v, t) \, dv$ is constant in space and time.

For the following arguments the homogeneous equation must have a one-parameter family of stationary distributions $f_e(\rho, v)$ depending only on the density, i.e., for ρ fixed we have

$$f(v, t) \to f_e(\rho, v) \qquad \text{for} \quad t \to \infty$$

no matter what the initial distribution of the homogeneous equation is. For example, for the special kinetic model in Section 5 this is numerically verified in ref. 16.

Depending on the family of stationary distributions, one defines the following equilibrium quantities: The mean velocity

$$u_e(\rho) = \frac{1}{\rho} \int_0^w v f_e(\rho, v) \, dv \tag{6}$$

the traffic pressure

$$p_e(\rho) = \int_0^w (v - u_e(\rho))^2 \, f_e(\rho, v) \, dv \tag{7}$$

the Enskog or anticipation coefficient

$$a_e(\rho) = \int_{[0, w]^2} \left[v_1 - \int_0^w v \sigma(v, v_1, v_2; \rho) \, dv \right]$$
$$\times H(v_1, v_2) \, |v_1 - v_2| \, k(H(v_1, v_2), \rho) \, f_e(\rho, v_1) \, \partial_\rho f_e(\rho, v_2) \, dv_2 \, dv_1 \tag{8}$$

and the interaction frequency

$$v_e(\rho) = \frac{1}{\rho} \int_{[0, w]^2} |v_1 - v_2| \, k(H(v_1, v_2), \rho) \, f_e(\rho, v_1) \, f_e(\rho, v_2) \, dv_1 \, dv_2 \tag{9}$$

The last expression has to be corrected with a suitable term if passing is included; see Section 5.

4.2. Balance Equations

We start the derivation of fluid dynamic equations by multiplying the inhomogeneous kinetic equation (4) with the property $\phi(v)$ and integrating it with respect to v. One obtains the following set of balance equations:

$$\partial_t \int_0^w \phi f \, dv + \partial_x \int_0^w v \phi f \, dv = \int_0^w \phi(v) \, C^+(f)(x, v, t) \, dv$$

We define the density n_ϕ of the property ϕ as

$$n_\phi = \int_0^w \phi f \, dv$$

The important point in deriving fluid dynamic equations from kinetic Enskog equations is to identify clearly the flux and the source terms in the equation. The flux of ϕ due to the kinetic advection part is as usual

$$q_\phi = \int_0^w v\phi f \, dv$$

However, there is a second contribution to the flux coming from the Enskog collision term due to the finite size of the interaction thresholds. To obtain this flux we separate the Enskog interaction term into the interaction term of the homogeneous equation and the deviation from the homogeneous term:

$$C^+ = C - (C - C^+)$$

We mention that we do not proceed here exactly as in Enskog's theory of a dense gas. The fact that the velocities are only positive requires a slightly different treatment. Rewriting the balance equations, we get then

$$\partial_t n_\phi + \partial_x q_\phi + E_\phi = S_\phi$$

with Enskog flux term

$$E_\phi = \int_0^w \phi(v)[\, C(f)(x, v, t) - C^+(f)(x, v, t)\,] \, dv$$

and source term

$$S_\phi = \int_0^w \phi(v) \, C(f)(x, v, t) \, dv$$

Using $\phi(v) = 1$ and $\phi(v) = v$, one obtains equations for the traffic flow density

$$n_1 = \rho = \int_0^w f \, dv$$

and the traffic flux

$$n_r = \rho u = \int_0^w vf \, dv$$

$u = n_r/n_1$ denotes the mean velocity. Since $S_1 = 0$ and $E_1 = 0$, we get for $\phi(v) = 1$ the continuity equation

$$\partial_t \rho + \partial_x(\rho u) = 0$$

Moreover, for $\phi(v) = v$ the acceleration equation

$$\partial_t(\rho u) + \partial_x(p + \rho u^2) + E = S$$

is obtained with

$$p = \int_0^w (v - u)^2 f \, dv$$

$$E = E_r = \int_0^w v[\, C(f)(x, v, t) - C^+(f)(x, v, t)\,] \, dv$$

and

$$S = S_r = \int_0^w vC(f)(x, v, t) \, dv$$

To obtain closed equations for ρ and u one has to specify the dependence of p, S, and E on ρ and u.

4.3. Closure Relations

There are a variety of possible closure relations which could be borrowed from gas dynamics. We restrict ourselves here to the derivation of nonviscous fluid dynamic equations. As usual, to find closure relations for the balance equations one has to use the stationary solutions of the homogeneous kinetic equation (5). All parameters of the fluid dynamic equation can be identified from these solutions. In the following $f_e(\rho, v)$ is the local equilibrium distribution associated to the local density $\rho = \rho(x, t)$.

We set

$$p = \int_0^w (v - u)^2 f \, dv \sim \int_0^w (v - u_e(\rho))^2 f_e(\rho, v) \, dv = p_e(\rho)$$

with p_e defined in (7). The Enskog term

$$E = \int_0^w v[\, C(f)(x, v, t) - C^+(f)(x, v, t)\,]\, dv$$

$$= \int_{[0,\, w]^2} \left[v_1 - \int_0^w v\sigma(v, v_1, v_2; \rho)\, dv \right] |v_1 - v_2|\, k(H(v_1, v_2), \rho)$$

$$\times [\, f(x, v_1, t)\, f(x + H(v_1, v_2), v_2, t) - f(x, v_1, t)\, f(x, v_2, t)\,]\, dv_2\, dv_1$$

is approximated in the following way:

We linearize and substitute the stationary distributions f_e for f. Using the definition of $a_e(\rho)$ in (8), this yields

$$E \sim \int_{[0,\, w]^2} \left[v_1 - \int_0^w v\sigma(v, v_1, v_2; \rho)\, dv \right]$$

$$\times H(v_1, v_2)\, |v_1 - v_2|\, k(H(v_1, v_2), \rho)\, f_e(\rho, v_1)\, \partial_\rho f_e(\rho, v_2)\, \partial_x \rho\, dv_2\, dv_1$$

$$= a_e(\rho)\, \partial_x \rho$$

The source term is treated as, for example, in semiconductor theory (see, e.g., ref. 26) by using a relaxation-time approximation:

$$S \sim - \int_0^w v \frac{f - f_e}{T_e(\rho)}\, dv = - \frac{\rho u - \rho u_e(\rho)}{T_e(\rho)}$$

with

$$T_e(\rho) = \frac{1}{v_e(\rho)}$$

where v_e and u_e are defined in (9) and (6), respectively. All together this yields fluid dynamic equations of the form

$$\partial_t \rho + \partial_x(\rho u) = 0$$

$$\partial_t(\rho u) + \partial_x(p_e(\rho) + \rho u^2) + a_e(\rho)\, \partial_x \rho = \frac{\rho u_e(\rho) - \rho u}{T_e(\rho)}$$

where the quantities $p_e(\rho)$, $a_e(\rho)$, $u_e(\rho)$, and $T_e(\rho)$ are determined from the stationary solution of the homogeneous kinetic equations. The obtained equation looks formally similar to Payne's equation.[5] However, we have obtained in the above way coefficients that are derived from microscopic considerations, not only constants that are fitted to measured data.

Remark 1. We remark that the term $a_c(\rho) \, \partial_x \rho$ is not due to the usual kinetic pressure term, but only to the Enskog correction. The influence of the kinetic pressure is given by p_c.

Remark 2. Obviously the procedure can be extended without difficulties to get, for example, a third equation for the variance. See ref. 8 for such an equation. For the derivation of equations with viscous terms as in ref. 6 a more complicated treatment is required.

5. NUMERICAL RESULTS

In this section we present a numerical solution of the kinetic equation for a special model in a space-inhomogeneous situation. Moreover, we evaluate explicitly the coefficients derived in the last section from the homogeneous kinetic equation. The macroscopic equations with these coefficients are solved and the results are compared to the kinetic solution.

5.1. Kinetic Model

Before describing the discretization scheme for the kinetic equation we state briefly the explicit example for the kinetic model that we use for the simulation (see ref. 16 for more details). Consider Eq. (2) with two thresholds ($N = 2$), one for slowing-down and one for acceleration interactions. The slowing-down threshold is given by a headway

$$H_1(v_1, v_2) = h$$

where h is a positive constant. For slowing down Ω_1 is given by

$$\Omega_1 = \{(v_1, v_2) \in [0, w]^2; v_1 > v_2\}$$

The slowing-down term is modeled by a Prigogine-like term. However, we do not restrict the slowing down to the actual speed of the leading vehicle as in the Prigogine work, but to a range of speeds smaller than this one. We allow slowing down of car 1 to a velocity $v \in [\beta v_2, v_2]$, where β is some positive constant smaller than 1. In this range a uniform distribution of velocities is assumed due to the lack of more precise knowledge.

If the following car is faster than the leading one, we have, alternatively to slowing down, also to take into account the possibility of passing with a certain probability P analogous to the Prigogine model in

Section 2. We assume the velocity of the passing car to remain the same as before. This gives

$$\sigma_1(v, v_1, v_2, \rho) = P\delta(v_1 - v) + (1 - P) \frac{1}{v_2(1 - \beta)} \chi_{[\beta v_2, v_2]}(v)$$

where δ denotes the delta function and $\chi_{[a, b]}$ is the characteristic function of the interval $[a, b]$.

The probability of passing P depends on $\rho(x, t)$ like

$$P = 1 - \frac{\rho}{\rho_m}$$

where ρ_m denotes the maximal density, which is given by the number of lanes. This is the same dependence as in the Prigogine model. ρ_m is a location-dependent road condition parameter.

The acceleration threshold is assumed to be given by the same constant

$$H_2(v_1, v_2) = h$$

For a more consistent microscopic model one would have to choose this constant larger than the constant of the slowing-down threshold. However, for the kinetic model the limit case of equal constants does not yield any problems. For acceleration Ω_2 is given by

$$\Omega_2 = \{(v_1, v_2) \in [0, w]^2; v_1 < v_2\}$$

We assume that car 1 accelerates from its actual speed v_1 to a certain range of speeds between v_1 and $v_1 + \alpha(w - v_1)$. Moreover, we assume again, due to the lack of more precise knowledge, that the velocity after acceleration is uniformly distributed in the range $[v_1, v_1 + \alpha(w - v_1)]$. This leads to the distribution function

$$\sigma_2(v, v_1, v_2; \rho) = \frac{1}{\alpha(w - v_1)} \chi_{[v_1, v_1 + \alpha(w - v_1)]}(v)$$

Since for dense traffic the possibility of acceleration is more restricted than for traffic flow with a low density, α is supposed to depend on the density $\rho(x, t)$ in the following way:

$$\alpha = \alpha_0 \left(1 - \frac{\rho}{\rho_m}\right)$$

where $\alpha_0 \leqslant 1$ is some positive constant. Note that this model fulfills condition (3). Using the notation at the beginning of Section 4, one can define σ by σ_1 and σ_2. Since a completely satisfying correlation model is not available, we assume for the simulation that $k(h, \rho)$ is equal to 1.

This model is now numerically investigated.

We treat the kinetic equation by a discretization scheme as follows: A simple standard discretization of the equation in velocity space needs a large number of discretization points in order to describe correctly the influence of the singularities appearing at $v = 0$ and $v = w$. Therefore we divide the velocity space into a certain number of cells and calculate the transition rates between the cells given by the above kinetic equation. Setting $w = 1$, we introduce gridpoints

$$x_i = i \, \Delta x, \quad i = 0,..., M, \quad \Delta x = \frac{L}{M}, \quad v_j = \frac{j + 1/2}{N}, \quad j = 0,..., N - 1$$

and the values $f_{i,j} = f(x_i, v_j, t)$. The discretized Enskog equation is then given by

$$\partial_t f_{i,j} + v_j \frac{f_{i,j} - f_{i-1,j}}{\Delta x}$$

$$= \frac{1}{N^2} \sum_{k,l=0}^{N-1} \bar{\sigma}_{i,j,k,l} |v_k - v_l| f_{i,k} \tilde{f}^h_{i,l} - \frac{1}{N} f_{i,j} \sum_{k=0}^{N-1} |v_j - v_k| \tilde{f}^h_{i,k}$$

with

$$\tilde{f}^h_{i,j} = (1 - \alpha) f_{i + \bar{i},j} + \alpha f_{i + \bar{i} + 1,j}$$

where

$$\bar{i} = \max\left(i: i < \frac{h}{\Delta x}\right), \qquad \alpha = \frac{h}{\Delta x} - \bar{i}$$

and

$$\bar{\sigma}_{i,j,k,l} = N^3 \int_{(j,k,l)/N}^{(j+1,k+1,l+1)/N} \sigma(v, v_1, v_2; \rho_i) \, dv \, dv_1 \, dv_2$$

where $\bar{\sigma}_{i,j,k,l}$ is evaluated analytically. The density used to compute σ is calculated by

$$\rho_i = \frac{1}{N} \sum_{j=0}^{N-1} f_{i,j}$$

The time derivative is treated in a standard way. This means that we implement a discrete-velocity approximation of the kinetic equation using the discrete transition rates $\bar{\sigma}_{i,j,k,l}$ which are averaged values for each cell. The most important fact about this discretization is that the discrete version of the continuity equation is automatically fulfilled, since

$$\frac{1}{N} \sum_{j=0}^{N-1} \bar{\sigma}_{i,j,k,l} = 1$$

The scheme needs only a very low number—approximately ten cells—in velocity space to yield resonable results.

5.2. Fluid Dynamic Model

The fluid dynamic equation is solved with a splitting algorithm treating the flux and the relaxation part in separate steps. The flux part, as usual, is written in conservative form and then treated by a conservative scheme. We have simply chosen the Lax–Friedrichs algorithm.

To obtain the coefficients for the fluid dynamic equations we have to compute the stationary distributions of the homogeneous kinetic equation (5). In the present case the interaction frequency leading to a relaxation to the stationary distribution must be corrected, as already mentioned in Section 4, due to the inclusion of the possibility of passing. We get

$$v_e(\rho) = \frac{1}{\rho} \int_{[0,\,w]^2} |v_1 - v_2|\, k(H(v_1, v_2), \rho)\, f_e(\rho, v_1)\, f_e(\rho, v_2)\, dv_1\, dv_2$$

$$- \frac{P(\rho)}{\rho} \int_{\Omega_1} |v_1 - v_2|\, k(H(v_1, v_2), \rho)\, f_e(\rho, v_1)\, f_e(\rho, v_2)\, dv_1\, dv_2$$

To solve the homogeneous equation, we use a scheme similar to the one described in ref. 16. To obtain a high accuracy for the coefficients, an adaptive discretization of the velocity space is introduced additionally. The parameters in the kinetic model are chosen as $\alpha_0 = 0.3$ and $\beta = 0.3$. Figures 1–4 show plots of the quantities $u_e(\rho)$, $p_e(\rho)$, $v_e(\rho)$, and $a_e(\rho)$. The plot of the mean velocity fits quantitatively to measured data reported in ref. 27. At this point one should mention that the obtained coefficients depend on the microscopic model via the kinetic equation. They cannot be supposed to be the correct ones for any traffic flow situation.

5.3. An Inhomogeneous Traffic Flow Situation

In the following series of figures an inhomogeneous traffic flow situation is shown. We consider a highway with a reduction of the number of
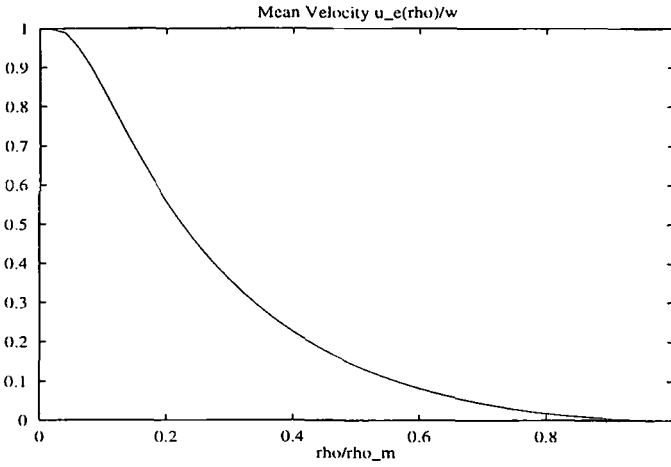
Fig. 1.   Mean velocity.

lanes from three to two after two-thirds along the highway under consideration. The example is calculated with the kinetic equation and the derived fluid dynamic equations with the coefficients computed above. The reduction of the lanes is simply achieved by changing the maximal density ρ_m.

The units in the following calculations are given by setting the maximal velocity w equal to 1 and the bumper-to-bumper distance h_0 equal
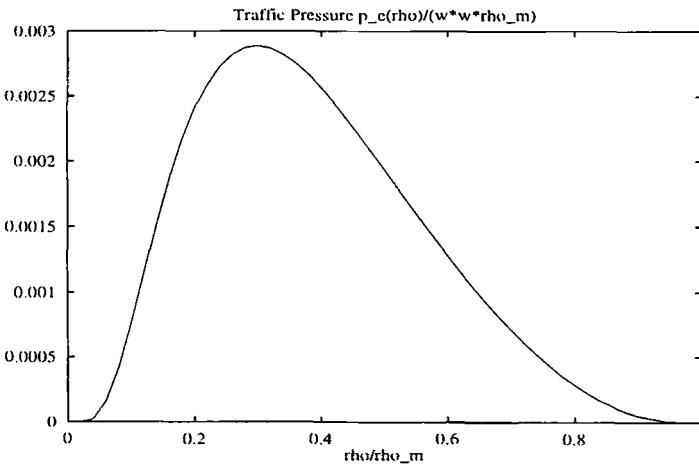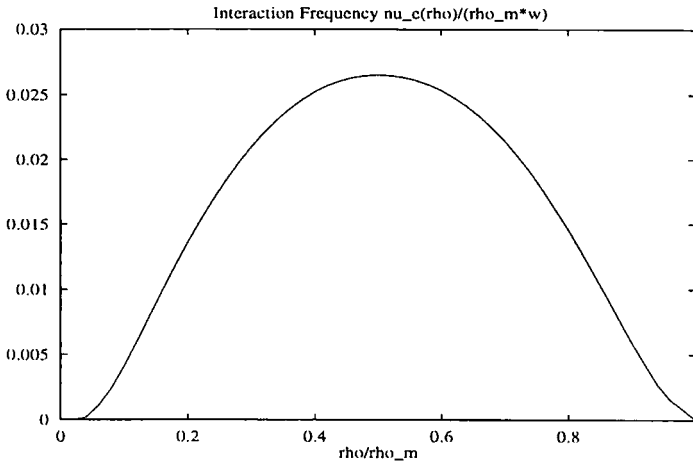

Fig. 2.   Traffic pressure.

Fig. 3. Interaction frequency.

to 1. This yields the time unit to be equal to the bumper-to-bumper distance divided by the maximal velocity, and the maximal density ρ_m equal to the number of lanes under consideration. The length L of the highway under consideration is equal to 1000. The space and time discretizations are $\Delta x = 1$ and $\Delta t = 0.5$ for the kinetic equation and $\Delta x = 0.5$ and $\Delta t = 0.125$ for the fluid dynamic equation. For the kinetic equation the number of points for the velocity discretization is chosen as $N = 40$ and the threshold value as $h = 5$.
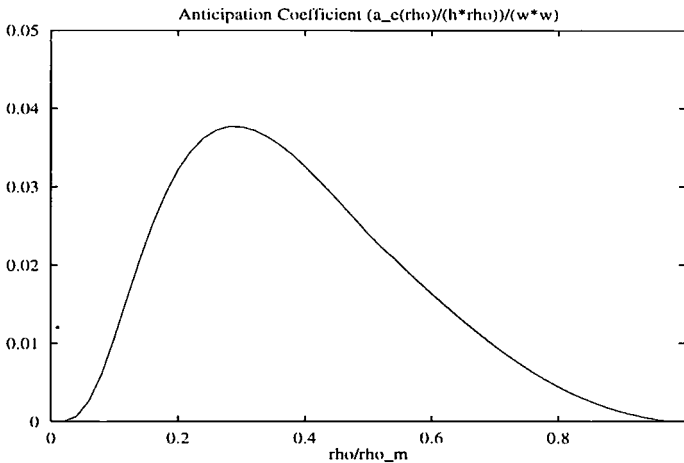
Fig. 4. Anticipation coefficient.

F(x,v)
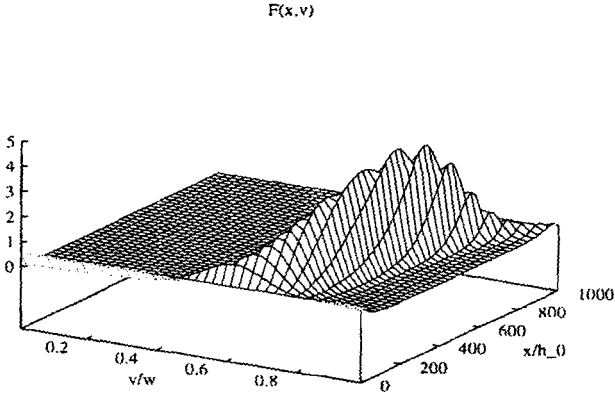
Fig. 5.   Space–speed distribution for $t = 1000$.

We start with an "empty" highway and prescribe the incoming distribution function for the kinetic equation. A constant number of incoming vehicles is used. The solution of the kinetic equation, i.e., the space–speed distribution of the vehicles, is shown for different times in Figs. 5–7. Starting with an empty highway, one observes in Fig. 5 free flow of the vehicles until the stretch is completely filled with vehicles. The overall density is small compared to the maximal density, such that there is no influence of the bottleneck. When the stretch finally is filled with vehicles, the speed distribution drops at the bottleneck, reflecting the formation of a traffic jam, as seen in Fig. 6. Two modes of the distribution function appear at the beginning of the bottleneck. In Fig. 7 the jam is running
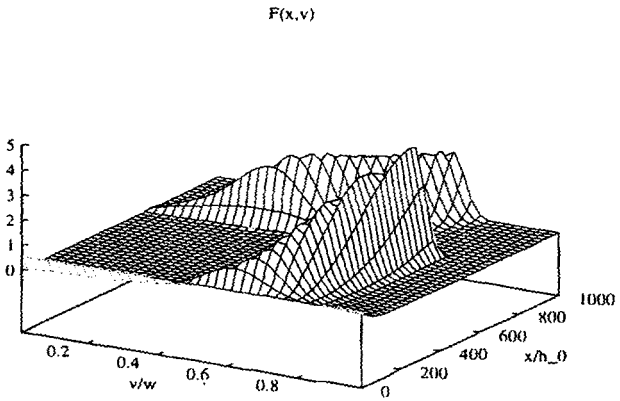
F(x,v)

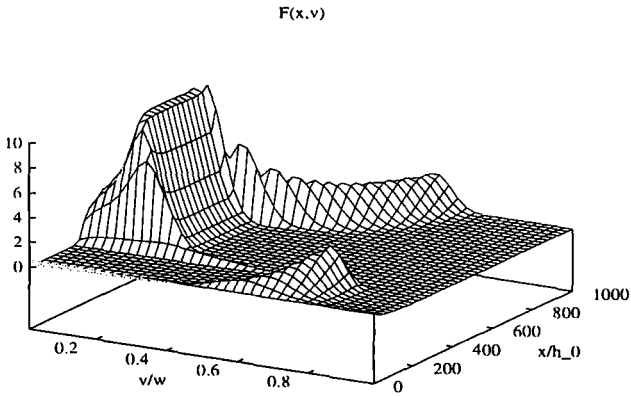Fig. 6.   Space–speed distribution for $t = 2000$.

F(x.v)

Fig. 7.   Space–speed distribution for $t = 8000$.

backward. In the bottleneck itself the speed distribution is recreated, showing normal traffic flow with lower density.

Figures 8 and 9 show the time development of the density of vehicles for the fluid dynamic and kinetic equations.

We mention that, as expected, both equations show very similar behavior. However, in the kinetic simulation the shocks are smeared out much more than in the fluid dynamic one. This is similar to the situation in the kinetic theory of gases.
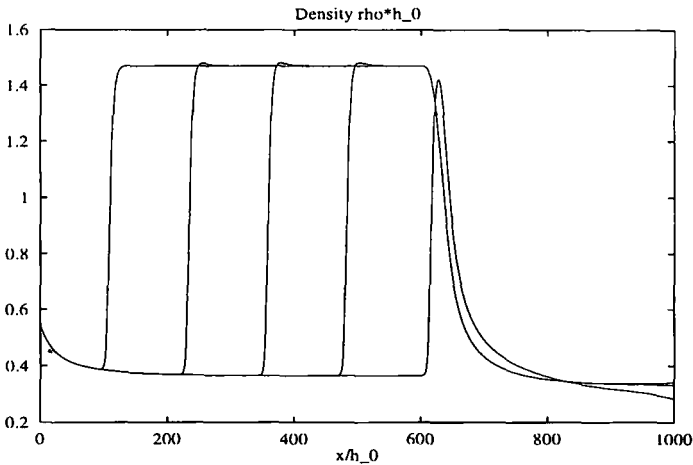

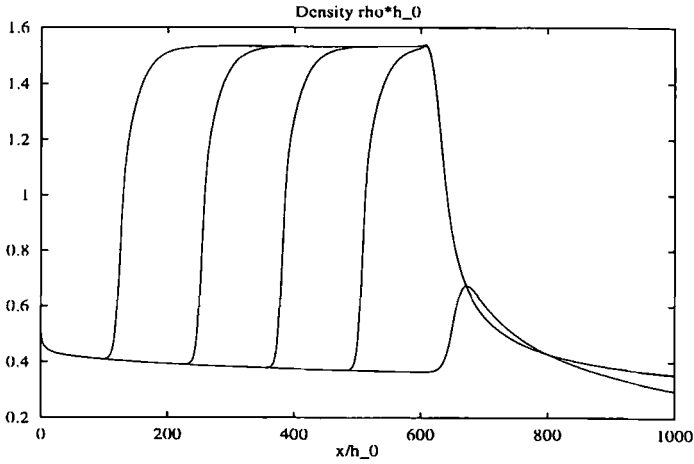Fig. 8.   Time development of the density for the fluid dynamic equation at $t = 2000$, 4000, 6000, 8000, 10,000.

Density rho*h_0

Fig. 9.   Time development of the density for the kinetic equation at $t = 2000$, 4000, 6000, 8000, $10,000$.

Finally, Fig. 10 shows the influence of the Enskog factor. The density of the kinetic solution with $h = 0$ (no Enskog correction) is plotted at time $t = 2300$. The case $h = 5$ is shown for comparison. In particular, one observes the explosion of the solution of the kinetic equations without the Enskog correction shortly after the time $t = 2300$, and no backward-running traffic jam is observed. This is in accordance with the considerations in Section 2.
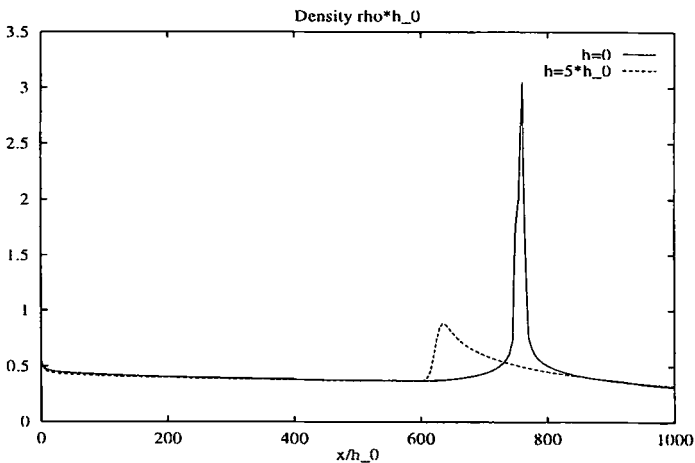
Density rho*h_0

Fig. 10.   Density for the kinetic equation without Enskog correction.

ACKNOWLEDGMENTS

REFERENCES

1. D. C. Gazis, R. Herman, and R. Rothery, Nonlinear follow-the-leader models for traffic flow, *Operations Res.* 9:545 (1961).
2. R. Wiedemann, Simulation des Strassenverkehrsflusses, Schriftenreihe des Instituts für Verkehrswesen der Universität Karlsruhe (1974).
3. D. L. Gerlough and M. J. Huber, Traffic Flow Theory, Transportation Research Board Special Report 1965, Washington, D.C. (1975).
4. G. B. Witham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
5. H. J. Payne, *Transport. Res. Rec.* 722:68 (1979).
6. R. D. Kühne, Macroscopic freeway model for dense traffic, in *9th International Symposium on Transportation and Traffic Theory*, N. Vollmuller, ed. (1984), p. 21.
7. B. S. Kerner and P. Konhäuser, *Phys. Rev. E* 50:54 (1994).
8. D. Helbing, Improved fluid dynamic model for vehicular traffic, *Phys. Rev. E* 51:3164 (1995).
9. I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (Elsevier, New York, 1971).
10. W. F. Phillips, Kinetic Model for Traffic Flow, National Technical Information Service, Springfield, Virginia (1977).
11. S. L. Paveri-Fontana, On Boltzmann like treatments for traffic flow, *Transport. Res.* 9:225 (1979).
12. D. Helbing, Gas-kinetic derivation of Navier–Stokes-like traffic equation. Preprint, University of Stuttgart (1995).
13. I. Prigogine and F. C. Andrews, A Boltzmann like approach for traffic flow, *Operations Res.* 8:789 (1960).
14. M. Lampis, *Transport. Sci.* 12:16 (1978).
15. P. Nelson, A kinetic model of vehicular traffic and its associated bimodal equilibrium solutions, *Transport. Theory Stat. Phys.* 24:383 (1995).
16. R. Wegener and A. Klar, A kinetic model for vehicular traffic derived from a stochastic microscopic model, *Transport. Theory Stat. Phys.* 25:785 (1996).
17. A. Klar, R. D. Kühne, and R. Wegener, Mathematical models for vehicular traffic, *Surv. Math. Ind.* 6:215 (1996).
18. K. Nagel and A. Schleicher, *Parallel Computing* 20:125 (1994).
19. M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, *Phys. Rev. E* 51:2939 (1995).
20. N. Anstett, Entwicklung eines ereignisorientierten Fahrzeug-Folge-Modells zur mikroskopischen Verkehrssimulation, Diplomarbeit Universität Stuttgart/Daimler Benz AG (1992).
21. C. Cercignani, *The Boltzmann Equation and its Applications* (Springer, New York, 1988).
22. J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972).

23. C. Cercignani and M. Lampis, On the kinetic theory of a dense gas of rough spheres, *J. Stat. Phys.* **53**:655–672 (1988).
24. J. Lebowitz, J. Percus, and J. Sykes, *Phys. Rev.* **188**:487 (1967).
25. P. Resibois, H-theorem for the (modified) nonlinear enskog equation, *J. Stat. Phys.* **19**:593 (1978).
26. P. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations* (Springer, New York, 1990).
27. R. D. Kühne and S. Rödiger, In *Proceedings of the 1991 Winter Simulation Conference, Phoenix, Arizona,* B. L. Nelson, W. D. Kelton, and G. M. Clark, eds. (IEEE, Piscataway, New Jersey, 1991).